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LETTER TO THE EDITOR

Solution of a random chain problem: an approach using canonical variables of an integrable system

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Abstract. The correspondence between a one-dimensional tight binding Hamiltonian and the Lax eigenvalue problem of the classical Toda lattice is applied to a disordered chain. A model with specific diagonal and off-diagonal disorder is solved by using canonical spectral variables. An exact analytical expression for the density of states is presented.

We consider a one-dimensional tight binding Hamiltonian of the form

$$H=\sum_n [|n\rangle\varepsilon_n\langle n|+V_n(|n\rangle\langle n+1|+|n+1\rangle\langle n|)]. \tag{1}$$

The corresponding discrete Schrödinger equation for the wavefunction $\phi^E(n)$ is

$$(\varepsilon_n-E)\phi^E(n)+V_{n-1}\phi^E(n-1)+V_n\phi^E(n+1)=0 \qquad n=0,\pm1,\pm2,\ldots. \tag{2}$$

We want to study the distribution of its eigenvalues for random matrix elements ε, V taken from a probability distribution P:

$$\mathrm{d}P=p(\ldots,\varepsilon_1,\ldots,V_1,\ldots)\prod_k(\mathrm{d}\varepsilon_k\,\mathrm{d}V_k).$$

A direct way of doing this would be a variable transformation to eigenvalues and a calculation of their joint probability density. Clearly this seems in general a formidable task involving complicated Jacobians.

Unexpectedly, such a direct solution—at least for certain weight functions p—can be found in the field of non-linear classical Hamiltonian systems. There it has been shown (see, for example, Eilenberger 1981) that a variety of non-linear evolution equations can be written as the time evolution of a linear operator involving the dynamical variables as matrix elements. The spectrum of this operator—often called the Lax operator (Lax 1968) of the non-linear system—provides new coordinates for the system, allowing for a complete integration of the equations of motion.

However, the main result of this 'inverse spectral method' (Ablowitz *et al* 1974) relevant to our work is that the mapping from matrix elements to 'spectral data' appears as a classical canonical transformation (McLaughlin 1975), thereby preserving the volume elements in phase space.

The operator H in (1) can be interpreted as the Lax operator of the classical Toda lattice if we set $V_n=\exp[\frac{1}{2}(Q_n-Q_{n+1})]$, where Q_n and ε_n are displacements and momenta of the nth Toda particle.

The Toda lattice is an integrable Hamiltonian system with Poisson brackets

$$\{f,g\}=\sum_n\left(\frac{\partial f}{\partial Q_n}\frac{\partial g}{\partial \varepsilon_n}-\frac{\partial f}{\partial \varepsilon_n}\frac{\partial g}{\partial Q_n}\right) \tag{3}$$

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and Hamilton function

$$h_{\text{Toda}} = \sum_n (\tfrac{1}{2}\varepsilon_n^2 + V_n^2). \tag{4}$$

For the periodic $(N+1)$-particle Toda lattice of length L

$$\varepsilon_{n+N+1} = \varepsilon_n \tag{5a}$$

$$Q_{n+N+1} = Q_n + L \tag{5b}$$

it was shown (Flaschka and McLaughlin 1976, Sklyanin 1985) that the zero-boundary eigenvalues $E_1, \ldots, E_N$ of H defined by $\phi^{E_i}(0) = \phi^{E_i}(N+1) = 0$ satisfy $\{E_i, E_j\} = 0$ for $i, j = 1, \ldots, N$.

Variables canonically conjugate to the E are given by $f_i = \ln|\phi^{E_i}(N+2)|$ where $\phi^{E_i}(1) = 1$ was taken. We therefore have the relations

$$\begin{aligned} \{E_i, E_j\} &= \{f_i, f_j\} = 0 \\ \{f_i, E_j\} &= \delta_{ij}. \end{aligned} \tag{6}$$

The $2N$ spectral variables E_i, f_i, $i = 1, \ldots, N$ together with ε_{n+1}, Q_{N+1} form a complete set of canonical variables.

We use the Liouville volume element $\mathrm{d}\Omega$ in phase space to build up the probability distribution P for the matrix elements ε_i, V_j. Since $\mathrm{d}\Omega$ is invariant under canonical transformations the Jacobian equals one and we simply have

$$\mathrm{d}\Omega = \prod_{i=1}^{N} (\mathrm{d}Q_i\, \mathrm{d}\varepsilon_i) = \prod_{i=1}^{N} (\mathrm{d}f_i\, \mathrm{d}E_i). \tag{7}$$

Here we have fixed $Q_{N+1} = \varepsilon_{N+1} = 0$ for convenience.

To define a statistical weight function we have to note that, in principle, every symmetric function of ε and V can be expressed by spectral variables too. But up to now this construction has been carried out explicitly only for the function h_{Toda} (4).

We define

$$\mathrm{d}P \sim \mathrm{d}\Omega\, \mathrm{e}^{-h} \tag{8}$$

with

$$h = h_{\text{Toda}} = \sum_{n=1}^{N+1} (\tfrac{1}{2}\varepsilon_n^2 + V_n^2).$$

It can be shown (Flaschka and McLaughlin 1976, Toda 1981) that this function is expressed in terms of the spectral variables via

$$h_{\text{Toda}} = \sum_{i=1}^{N} \left(\tfrac{1}{2}E_i^2 + 2\exp(-\tfrac{1}{2}L) \frac{\cosh f_i}{\Pi_{k,k\neq i}|E_i - E_k|} \right). \tag{9}$$

The weak correlations of the random variables V_i in (8) due to the constraint (cf 5(b)) $L = -2\sum_{n=1}^{N+1} \ln V_i$ can be neglected in the thermodynamic limit $N \to \infty$. Using the equivalent constant pressure ensemble $\mathrm{d}P_\gamma$ they can be explicitly avoided:

$$\mathrm{d}P_\gamma \sim \mathrm{d}\Omega \int_{-\infty}^{\infty} \mathrm{d}L\, \mathrm{e}^{-h}\, \mathrm{e}^{-\gamma L} \qquad \gamma > 0. \tag{10}$$

L and γ are related by $\gamma = \partial \ln Z_L / \partial L$ with the partition function $Z_L = \int \mathrm{d}\Omega\, \mathrm{e}^{-h}$.

We obtain, after simple variable transformations

$$dP_\gamma \sim \prod_{n=1}^{N} [d\varepsilon_n \, dV_n \, V_n^{2\gamma-1} \exp(-\tfrac{1}{2}\varepsilon_n^2 - V_n^2)]. \tag{11}$$

The Hamiltonian (1) with the same distribution of hopping terms V_n but with $\varepsilon_n \equiv 0$, $n = 0, \pm 1, \ldots$, has been studied by Dyson (see Dyson 1953, Theodorou and Cohen 1976). In our model *both* quantities are random variables.

The joint probability density $w(E_1, \ldots, E_N)$ of eigenvalues is easily obtained by substituting the classical Hamilton function (9) into (8) and integrating over the f. We obtain

$$w(E_1, \ldots, E_N) = Z_L^{-1} \exp\left(- \sum_{i=1}^{N} (\tfrac{1}{2}E_i^2 - \ln\{2K_0[2\exp(-N/l(E_i))]\}) \right). \tag{12}$$

K_0 is a modified Bessel function (Abramowitz and Stegun 1970) and the expression

$$1/l(E_i) = \frac{1}{N}\left(- \sum_{i=1}^{N+1} \ln V_i + \sum_{\substack{k \\ k \neq i}} \ln|E_i - E_k| \right)$$

equals the Thouless inverse localisation length (Thouless 1972) for an eigenstate of the Hamiltonian (1).

We next choose a large E interval $(-c, c)$, partition it into k intervals of length Δ and calculate the probability $W(n_1, \ldots, n_k)$ of finding the first n_1 eigenvalues in $(-c, -c+\Delta)$, the next n_2 eigenvalues in $(-c+\Delta, -c+2\Delta)$, etc, with $\Sigma_{i=1}^{k} n_i = N$. Integrating $w(E_1, \ldots, E_N)$ over the corresponding volume in the space of $E_1, \ldots, E_N$ we find

$$W(n_1, \ldots, n_k) = w(E_1', \ldots, E_N') \prod_{i=1}^{k} \left(\frac{\Delta^{n_i}}{n_i!} \right) \tag{13}$$

where $(E_1', \ldots, E_N')$ is some interior point of the region of integration.

For $N \to \infty$ the macroscopical eigenvalue distribution is found from maximising W with respect to n_i. To perform this calculation in the limit $\Delta \to 0$ (13) is expressed in terms of the density of states (e.g. $n/N \simeq \rho(E)\Delta$).

The inverse localisation length

$$1/l(E) = \frac{L}{2N} + \int_{-\infty}^{\infty} \rho(x) \ln|E - x| \, dx \tag{14}$$

is a positive quantity for the eigenvalue distribution of interest (all eigenstates are localised for one-dimensional disorder). Thus the approximation $K_0[2\exp(-N/l(E))] \approx N/l(E)$ is valid for $N \to \infty$.

We arrive at

$$W(n_1, \ldots, n_k) \to W[\rho] = \exp(-F[\rho]) \tag{15}$$

with

$$F[\rho] = N \int_{-\infty}^{\infty} dE \, \rho(E)[\tfrac{1}{2}E^2 + \ln(\tfrac{1}{2}\rho l) - 1]$$

where the limit $c \to \infty$ is implied. Note that l depends on ρ according to (14). Straightforward variation of $F[\rho] - \mu N \int \rho(E) \, dE$ leads to the integral equation:

$$\tfrac{1}{2}E^2 - \mu + \ln \omega(E) - 2 \int_{-\infty}^{\infty} \ln|E - x| \omega(x) \, dx = 0 \tag{16}$$

for the auxiliary function $\omega = \frac{1}{2}l(E)\rho(E)$. The chemical potential μ is introduced to satisfy the constraint $\int \rho(E)\,\mathrm{d}E = 1$. Differentiating (16) with respect to μ and comparing with (14) shows that

$$\rho(E) = \frac{L}{N}\frac{\partial}{\partial \mu}\omega(E).$$

Using $L/N = \partial\mu/\partial\gamma$ we finally have

$$\rho(E) = \frac{\partial}{\partial \gamma}\omega(E). \tag{17}$$

The non-linear equation (16) also appears in the classical Bethe ansatz treatment of the statistical mechanics of the Toda lattice. It has recently been solved analytically (Opper 1985). Substituting this solution into (17) yields our final expression for the density of states

$$\rho(E) = \pi^{-1}\frac{\partial^2}{\partial\gamma\,\partial E}\,\mathrm{Im}\ln\int_0^\infty \mathrm{d}t\,\exp(\mathrm{i}Et)\exp(-t^2/2)t^{\gamma-1}. \tag{18}$$

We have plotted this result for $\gamma = 5$ in figure 1 together with the ensemble-averaged density of states from a numerical simulation. Both calculations coincide.

To summarise: the density of states of a disordered chain with a specific diagonal as well as off-diagonal disorder has been calculated using the canonical structure of spectral variables.

Obviously an analogous procedure can be applied to the Lax eigenvalue problems of other integrable systems. Their statistical mechanics have been extensively studied using classical Bethe ansatz techniques (see, for example, Timonen *et al* 1986). These results may be used to obtain new exact solutions for disordered systems.

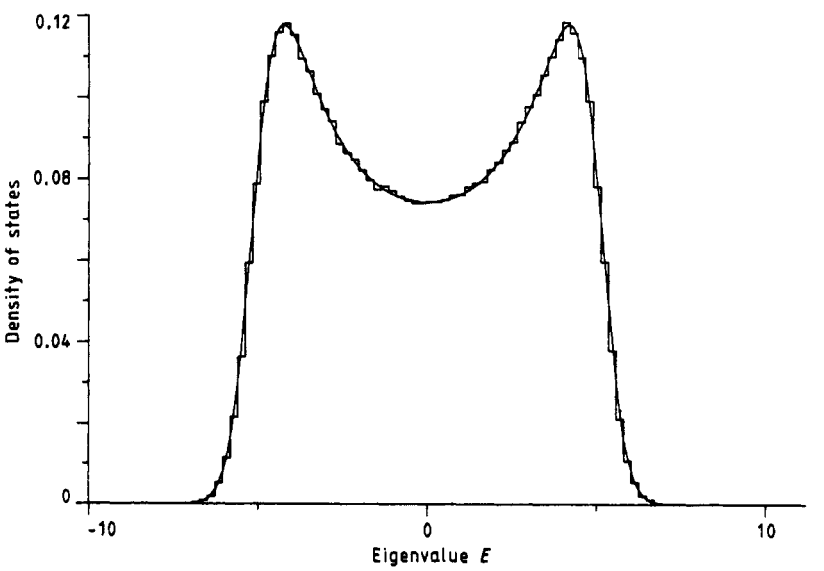

Figure 1. Density of states for $\gamma = 5$. Smooth curve: analytical solution (18). The step function was obtained by counting eigenvalues in intervals of length $\Delta = 0.2236$ and averaging over a sample of 15 chains ($N = 20\,000$).

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